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**Optimal collective pension investment**

## Overall goal

- ▶ What are the potential benefits of collective schemes?
- ▶ What are the pros and cons of current UK proposals such as the Royal Mail CDC scheme?
- ▶ How should collective pensions be run?
- ▶ What are the policy implications in the UK?

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- ▶ Joint work with the Pension Policy Institute
- ▶ Industry advisory board
- ▶ Catherine Donnelly (Heriot Watt), Cristin Buescu (KCL), James Dalby (KCL), Rohan Hobbs (KCL)

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## This talk

- ▶ Optimal collective pension investment in complete markets with idiosyncratic risk
- ▶ A very quick look at optimal collective pension investment with systematic risk

## Collective Drawdown Approach

- ▶ Each investor has a separate investment portfolio for accounting purposes
- ▶ They invest and withdraw according to their preferred strategy
- ▶ Funds of deceased investors are shared among survivors

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probability of dying over period  $\times$  wealth at end of period

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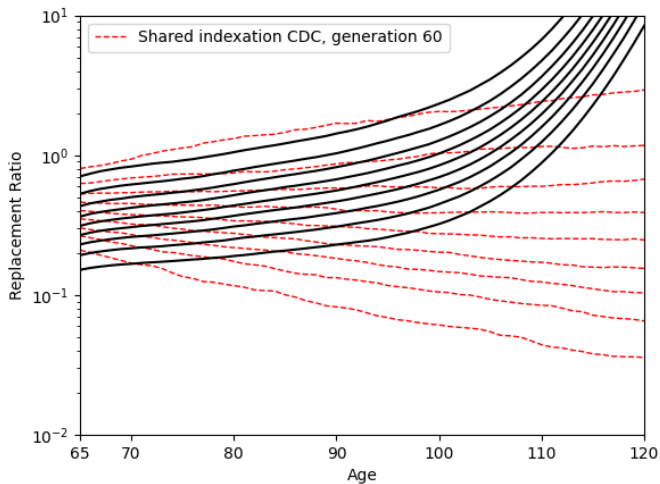
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## Claim:

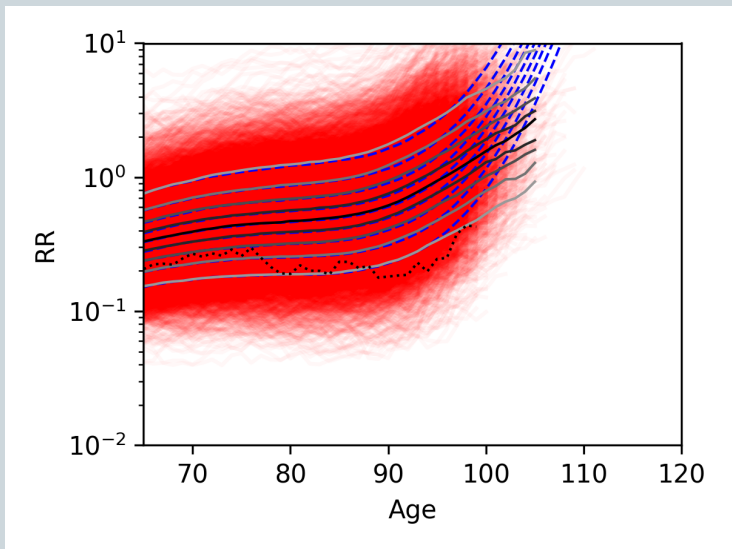
- ▶ If the market is complete then, subject to certain conditions, this is asymptotically optimal (if one additionally sets aside  $\epsilon$  in a riskless account to guarantee a lower bound)

DWP: “Evidence from other countries suggests that by sharing the risks among members, schemes providing ‘collective benefits’ may provide more stable outcomes than individual Defined Contribution schemes currently available in the UK. For example, while members are saving for retirement they can get some degree of protection from fluctuations in the financial markets”

# UK CDC design vs Optimum



## Finite fund size (20 per generation)



## When do mutually beneficial contracts exist in complete markets?

- ▶ If your maximum utility is already achieved, you may be willing to give away money. This includes a maximum utility of  $-\infty$ .
- ▶ If your preferences are time-inconsistent and you incorrectly believe your maximum utility will be achieved, you may be willing to give away money.
- ▶ If your preferences depend on the outcomes of others (either through altruism or jealousy) you may want to redistribute wealth
- ▶ If people can be compelled to join the scheme, a central planner might have their own notion of “optimal” and may want to redistribute wealth

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- ▶ If people can be compelled to join the scheme, a central planner might have their own notion of “optimal” and may want to redistribute wealth
- ▶ If there is an infinite time horizon, Ponzi type schemes may be possible
- ▶ If people can be compelled to join the scheme it may be possible to offer uncollateralized loans

## A deterministic model of shared-indexation CDC (or fully-funded DB)

- ▶ All individuals contribute an amount  $C(1+g)^t$  to a fund each year from the time they join a scheme to retirement  $n$  years later.
- ▶ Each year the same number of members join
- ▶ Each year, their current benefit entitlement grows to match inflation (which is constant and equal to  $i$ ) and they receive an additional benefit entitlement  $B(1+g)^t$ .
- ▶ Money invested in a riskless asset which grows at a rate  $r$ .
- ▶ The scheme has an infinite number of investors and there is no systematic longevity risk.
- ▶ The regulator insists assets=liabilities at all times.

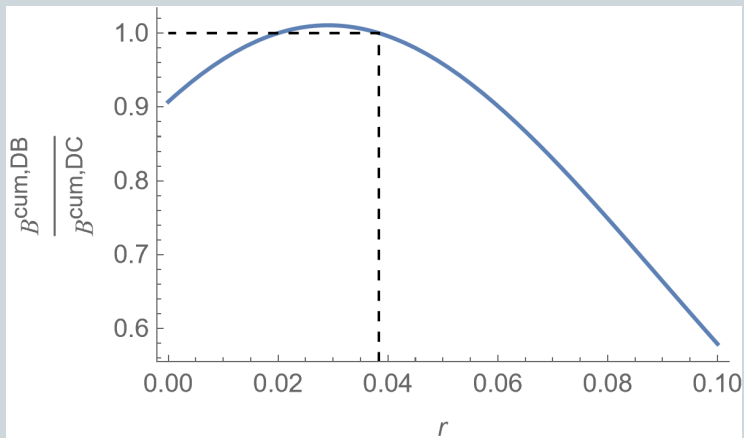
### Theorem

*The ratio of benefits received from such a scheme to the benefits received in a riskless DC + annuity scheme is*

$$\frac{(\alpha - 1)(i + 1)n\alpha^n(r - g) ((i + 1)^n - (g + 1)^n)}{(r + 1)(i - g) (\alpha^n - 1) ((r + 1)^n - (g + 1)^n)}.$$

*where  $\alpha = (1 + r)/(1 + i)$ . As CDC funds invest in risky assets, this should be seen as an estimate for the expected ratio for CDC funds.*

## Benefit ratio calibrated to OBR Figures



## Uncollateralized loans

- ▶ Time points  $\mathcal{T} = \{0, 1\}$
- ▶ Investors A and B have consumption processes  $(\gamma_t^A)_{t \in \mathcal{T}}, (\gamma_t^B)_{t \in \mathcal{T}}$
- ▶ Investor A has reward function  $\hat{\mathcal{J}}^A(\gamma^A) = \mathbb{E}((\gamma_1^A)^{1/2})$
- ▶ Investor B has utility function  $\hat{\mathcal{J}}^B(\gamma^B) = \mathbb{E}((\gamma_0^B)^{1/2})$

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- ▶ Investor A receives \$1 of income at time 0 and \$0 at time 1.
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- ▶ Investor A receives \$1 of income at time 0 and \$0 at time 1.
- ▶ Investor B receives \$0 of income at time 1 and \$1 none at time 0.
- ▶ If investor A loans their \$1 to B at time 0, B can consume it and still pay A back without harming their utility.
- ▶ The total fund wealth never drops below 0.
- ▶ But if B can renege on the deal, A will not make the loan.

## Formal problem setup (no idiosyncratic risk)

- ▶ Finite set of time points  $\mathcal{T} = \{0, 1, 2, \dots, T\}$
- ▶ Filtered probability space  $(\Omega^M, \mathcal{F}^M, (\mathcal{F}_t^M)_{t \in \mathcal{T}}, \mathbb{Q})$
- ▶ Finite set of individuals  $I$
- ▶  $(\eta_t^i)_{t \in \mathcal{T}}$  adapted process representing income of  $i \in I$

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- ▶  $(\gamma_{s,t}^i)_{t \in \mathcal{T}}$  adapted process representing financial position of  $i \in I$  at time  $s \in \mathcal{T}$
- ▶  $\Delta_t^i(\gamma) := \gamma_{t,t}^i$  consumption process of  $i$ .
- ▶  $\gamma_{s,t} = \gamma_{t,t}$  if  $t \leq s$

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- ▶  $\gamma_{s,t} = \gamma_{t,t}$  if  $t \leq s$
- ▶  $\hat{\mathcal{J}}_s^i$  maps adapted processes to an  $\mathcal{F}_s$ -measurable random variable
- ▶  $\hat{\mathcal{J}}_s^i(\Delta^i(\gamma)_t)$  determines preferences at time  $s$

# Admissible strategies

Given

- ▶  $J \subseteq I$ : the set of individuals under consideration
- ▶  $s \in \mathcal{T}$ : the time at which we propose a sequence of contracts to the members of  $J$ ,
- ▶  $b \in L^1((\mathbb{R}_{\geq 0})^J, \mathcal{F}_s^M, \mathbb{Q})$ : the non-negative budget of each individual
- ▶  $\bar{\gamma} \in L^1((\mathbb{R}_{\geq 0})^J, (\mathcal{F}_t^M)_{t \in \mathcal{T}_{<s}}, \mathbb{Q})$ : the currently agreed cashflows

An *admissible* strategy satisfies the *causality condition*

$$\gamma_{u,t}^i = \bar{\gamma}_t^i \quad \forall u \in \mathcal{T}_{\geq s}, t \in \mathcal{T}_{<u}, i \in J$$

and the *budget equation*

$$\sum_{i \in J} \left( b^i + \sum_{s \leq t \leq u} \eta_t^i - \sum_{s \leq t \leq T} \mathbb{E}_{\mathbb{Q}}(\gamma_{u,t}^i \mid \mathcal{F}_w^M) \right) \geq 0 \quad \forall u \in \mathcal{T}_{\geq s}, w \in \mathcal{T}_{\geq w}.$$

Write  $\mathcal{A}_s^J(b, \bar{\gamma})$  for the space of admissible strategies

# Acceptable strategies

Acceptable strategies are defined by backward recursion using the problem for a single investor  $J = \{i\} \subseteq I$

Given a tolerance  $\epsilon > 0$

$$\mathcal{G}_T^{\{i\}}(b, \bar{\gamma}, \epsilon) := \{\gamma \in \mathcal{A}_T^{\{i\}}(b, \bar{\gamma}) \mid \\ \hat{\mathcal{J}}_T^i(\Delta^i(\gamma)) \geq \hat{\mathcal{J}}_T^i(\Delta^i(\gamma')) \\ \forall \gamma' \in \mathcal{A}_T^{\{i\}}(b - \epsilon, \bar{\gamma})\}$$

$$\mathcal{G}_s^J(b, \bar{\gamma}, \epsilon) := \{\gamma \in \mathcal{A}_s^J(b, \bar{\gamma}) \mid \\ \hat{\mathcal{J}}_u^i(\Delta^i(\gamma)) \geq \hat{\mathcal{J}}_u^i(\Delta^i(\gamma')) \\ \forall i \in J, u \in \mathcal{T}_{>s}, \gamma' \in \mathcal{G}_u^{\{i\}}(b_u^i(\gamma) - \epsilon, \Delta^i(\gamma), \epsilon)\}$$

where

$$b_u^i(\gamma) := \sum_{u \leq t \leq T} \mathbb{E}_{\mathbb{Q}}(\gamma_{u-1,t}^i \mid \iota^\Omega \mathcal{F}_u^M)$$

## Remarks

- ▶ The tolerance  $\epsilon > 0$  is needed because the supremum in the single-investor problem might in principle not be achieved, in which case there may be no acceptable strategies with  $\epsilon = 0$ .
- ▶ Assume henceforth that  $\epsilon = 0$  and that the supremum is always achieved

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- ▶ We have not chosen any model for asset dynamics except implicitly through  $\mathcal{J}^i$ .
- ▶ An investor might have a probability model  $\mathbb{P}^i$  in mind and a utility function  $u_i$

$$\hat{\mathcal{J}}^i(\Delta^i(\gamma)) = \sum_{t \in \mathcal{T}} \mathbb{E}_{\mathbb{P}^i}(u_i(\Delta_t^i(\gamma)))$$

- ▶ They might also be a robust optimizer over a family of measures with a Epstein–Zin preferences etc.

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- ▶ They might also be a robust optimizer over a family of measures with a Epstein–Zin preferences etc.
- ▶ We are assuming a complete market assumption because we allow any integrable cashflow

## Definition

$\hat{\mathcal{J}}^i$  is *strictly monotone* if whenever  $s \in \mathcal{T}$ ,  $b, b' \in L^1(\mathbb{R}_{\geq 0}, \iota^\Omega \mathcal{F}_s^M, \mathbb{Q})$ ,  $b > b' \geq 0$ , and  $\bar{\gamma} \in L^1(\mathbb{R}_{\geq 0}, (\iota^\Omega \mathcal{F}_t^M)_{t \in \mathcal{T}}, \mathbb{Q})$  we can find

$$\gamma \in \mathcal{G}_s^{\{i\}}(b, \bar{\gamma})$$

with

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## Definition

$(\hat{J}_t)_{t \in \mathcal{T}}$  is *time consistent* if for all  $t \in \mathcal{T}$  and  $\gamma_1, \gamma_2 \in L^1(\Omega, (\mathcal{F}_t)_{t \in \mathcal{T}}, \mathbb{Q})$  one of the following  $\mathcal{F}_t$ -measurable events almost surely occurs:

- ▶  $\hat{J}_t(\gamma_1) > \hat{J}_t(\gamma_2)$  and  $\hat{J}_{t+1}(\gamma_1) > \hat{J}_{t+1}(\gamma_2)$ ;
- ▶  $\hat{J}_t(\gamma_1) \leq \hat{J}_t(\gamma_2)$  and  $\hat{J}_{t+1}(\gamma_1) \leq \hat{J}_{t+1}(\gamma_2)$ ;
- ▶ there exists  $t \in \mathcal{T}$ ,  $s \in \mathcal{T}_{\leq t}$  with  $(\gamma_1)_s \neq (\gamma_2)_s$ .

# No mutually beneficial contracts in complete markets

## Theorem

*If all preferences are time-consistent and strictly monotone then any acceptable strategy for a collective  $I$  yields the same value of  $\mathcal{J}_s^0$  as is attained in the strategy for a single individual.*

Proof: Backward induction.

## Incorporating idiosyncratic risk

- ▶ Handwaving suggests that if  $I$  indexes a type of investor and if we have an infinite number of investors of each type there will be no mutually beneficial contracts between different types.
- ▶ We'd like to show that collectives can be well approximated by infinite collectives

**Problem:** What does it mean to say that our preferences depend only upon our state and the market and the probabilities of different outcomes?

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**Problem:** What does it mean to say that our preferences depend only upon our state and the market and the probabilities of different outcomes?

- ▶ Standard definition of invariant preferences is that they are a function on the space of probability measures
- ▶ Attempting a definition in terms of random measures introduces unnecessary technicalities
- ▶ We will use invariance under transformations instead

## Definition

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## Definition

The canonical filtered probability space on  $\mathcal{T}$  is  $\prod_{t \in \mathcal{T}} [0, 1]$  with  $\mathcal{F}_t$  being the sigma algebra generated by functions which depend only on the first  $t$ -coordinates.

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## Theorem

(Corollary of Rokhlin's Theorem) If  $\Omega$  is a standard filtered probability space the  $\Omega \times \Omega^{\text{can}} \cong \Omega^{\text{can}}$  by a filtration preserving mod 0 isomorphism.

# Idiosyncratic preferences

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathcal{T}}, \mathbb{P})$  be a filtered probability space.

## Definition

An  $\Omega$  preference function at time  $s$  is a map

$$\hat{\mathcal{J}}_s : L^1(\mathbb{R}_{\geq 0}, (\mathcal{F}_t)_{t \in \mathcal{T}}, \mathbb{P}) \rightarrow L^1(\mathbb{R} \cup \{-\infty\}, \mathcal{F}_s, \mathbb{P})$$

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An invariant  $\Omega$  preference function is an  $\Omega \times \Omega^{\text{can}}$  preference function satisfying

$$\mathcal{J}_s(\gamma)(\omega, \omega^{\text{can}}) = \mathcal{J}_s(\gamma)(\omega, \phi(\omega^{\text{can}}))$$

for every filtration preserving mod 0 isomorphism of  $\Omega^{\text{can}}$ .

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An invariant preference function induces an  $\Omega \times \Omega'$  preference function on any filtered probability space  $\Omega'$  isomorphic to  $\Omega^{\text{can}}$ .

By Rokhlin's theorem, for any standard filtered probability space  $\Omega'$  we have a well-defined preference function  $\mathcal{J}_s$  on  $\Omega \times \Omega' \times \Omega^{\text{can}}$

# Probability set up

- ▶ Let  $\Omega^i$  be a discrete filtered probability space on  $\mathcal{T}$  representing the state of an individual
- ▶ Let  $\Omega^M$  be a filtered probability space on  $\mathcal{T}$  representing the market
- ▶ Let  $(\mathcal{J}_t^i)_{t \in \mathcal{T}}$  be  $\Omega^i \times \Omega^M$  be a set of invariant preferences

The total space for our problem is

$$\Omega := \Omega^M \times \prod_{i \in I} \prod_{\alpha \in \mathbb{Z}_{\geq 1}} \Omega^i \times \Omega^{\text{can}}$$

and is equipped with sub-algebras

- ▶  $\iota \mathcal{F}^M$  - market dependent processes
- ▶  $\iota \mathcal{F}^{i, \alpha}$  - processes which depend on the experience of the  $\alpha$ -th individual of type  $i$
- ▶  $\mathcal{F}_{s,t}^M := \iota(\mathcal{F}_t^M \times \prod_{i \in I} \prod_{\alpha \in \mathbb{Z}_{\geq 1}} \mathcal{F}^{i, \alpha}_t \times \mathcal{F}_t^{\text{can}})$  - cashflows available on the market at time  $s$

we also have

- ▶  $\mathcal{J}^{i, \alpha}$ , the preferences of individual  $(i, \alpha)$
- ▶  $\sigma_t^{i, \alpha}$ , the state of individual  $(i, \alpha)$  at time  $t$ , including their history.

## Permutations of the total space

Let  $S(\mathbb{Z}_{\geq 1})$  be the group of bijections of  $\mathbb{Z}_{\geq 1}$  to itself. Given

$$g \in G := \prod_{i \in I} S(\mathbb{Z}_{\geq 1})$$

we get a mod 0 isomorphism of  $\Omega$  permuting the individuals.

Let  $f^{i, \alpha, \beta}$  be the isomorphism swapping individuals  $(i, \alpha)$  and  $(i, \beta)$

**Definition:** For each  $(i, \alpha)$ , let  $\eta^{i,\alpha}$  be an income process which is  $\mathcal{F}^{i,\alpha}$ -measurable satisfying

$$\eta^{i,\alpha} = \eta^{i,\beta} \circ f^{i,\alpha,\beta}$$

Let  $\mathcal{I}$  be a finite set of pairs  $(i, \alpha)$ . An admissible solution given  $\eta$  and  $\mathcal{I}$  is a set of processes  $\gamma_{s,t}^{i,\alpha}$  which satisfy

- ▶ (causality)  $\gamma_{s,t}^{i,\alpha} = \gamma_{t,t}^{i,\alpha}$  if  $\{t < s\}$ ,
- ▶ (fairness)  $\gamma_{s,t}^{i,\alpha} = \gamma_{s,t}^{i,\beta}$  or  $\sigma_s^{i,\alpha} \neq \sigma_s^{i,\beta}$
- ▶ (measurability)  $C_{u,w}$  is  $\mathcal{F}_{u,w}$  measurable
- ▶ (budget)  $\mathbb{E}_{\mathbb{Q}}(C_{u,w}(\gamma) \mid \mathcal{F}_w) \geq 0$

where

$$C_{u,w}(\gamma) := \sum_{(i,\alpha) \in \mathcal{I}} \left( \sum_{0 \leq t \leq u} \eta_t^{i,\alpha} - \sum_{0 \leq t \leq T} \gamma_{u,t}^{i,\alpha} \right),$$

## Additional assumptions on preferences

- ▶  $\mathcal{J}_s^i$  is concave
- ▶  $\mathcal{J}_s^i$  is monotone:  $\mathcal{J}_s^i(\gamma_1) \geq \mathcal{J}_s^i(\gamma_2)$  if  $\gamma_1 \geq \gamma_2$
- ▶ Upper semi-continuous on closed convex sets containing  $\{\gamma \mid \mathcal{J}_s(\gamma) \geq b, \mathbb{E}_{\mathbb{P}} \mid \mathcal{F}_s(\gamma) \leq c\}$  in a chosen topology

**Example:**  $\Omega^i$  is generated by  $\tau_i$  a stopping time determining time of death.  $u$  is a continuous, increasing concave function.  $\mathbb{P}^{M,i}$  represents the beliefs of individuals of type  $i$  about the future evolution of the market.

$$\mathcal{J}_s^i(\gamma) = \mathbb{E}_{\mathbb{P}^i \times \mathbb{P}^{M,i}} \left( \sum_{t < \tau_i} u(\gamma_t) \right)$$

### Theorem

If  $\mathcal{J}_s$  is an invariant, concave, monotone  $\Omega^1$  preference function with the upper semi-continuity property then

$$\mathcal{J}_s(\mathbb{E}(\gamma \mid \mathcal{F}^{\Omega^1})) \geq \mathcal{J}_s(\gamma)$$

for all stochastic processes  $\gamma$  on  $\Omega^1 \times \Omega^{can}$

### Theorem

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$$\mathcal{J}_s(\mathbb{E}(\gamma \mid \mathcal{F}^{\Omega^1})) \geq \mathcal{J}_s(\gamma)$$

for all stochastic processes  $\gamma$  on  $\Omega^1 \times \Omega^{\text{can}}$

*Proof:*  $\Omega^{\text{can}}$  has a large transitive symmetry group. Choose a compact transitive subgroup which admits an invariant measure and use this to average over the symmetries. Apply an infinite-dimensional Jensen inequality to show that this increases  $\mathcal{J}$ . Use Fubini to show that this averaging is equivalent to taking an expectation.

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This implies that we may safely assume strategies between individuals all of type  $i$  are  $\iota \left( \left( \prod_{(i,\alpha)} \mathcal{F}^{i,\alpha} \right) \times \mathcal{F}^M \right)$  measurable. Hence we may assume this in our definition of an acceptable strategy.

## Infinite homogeneous fund problem

Given  $\gamma \in L^1(\mathbb{R}_{\geq 0}, \mathcal{F}_t^M, \mathbb{Q})$  define  $\hat{A}_s(\gamma)$  inductively by

$$\hat{A}_{-1}(\gamma) = L^1(\mathbb{R}_{\geq 0}, (\mathcal{F}_t)_{t \in \mathcal{T}}, \mathbb{Q})$$

and for  $t \in \mathcal{T}_{\geq 0}$

$$\hat{A}_t(\gamma) = \{\gamma' \in L^1(\mathbb{R}_{\geq 0}, (\mathcal{F}_t)_{t \in \mathcal{T}}, \mathbb{Q}) \mid \mathbb{E}_{\mathbb{Q}}(\gamma'_t \mid \mathcal{F}_t^M) = \gamma_t, \gamma' \in \operatorname{argmax}_{\gamma'' \in \hat{A}_{t-1}} \mathcal{J}_{t-1}^i(\gamma'')\}$$

(we must assume this exists).

Define

$$\hat{\mathcal{J}}_s^i(\gamma) = \sup_{\gamma \in \hat{A}_s(\gamma)} \mathcal{J}_s^i(\gamma).$$

# Limiting approach

- ▶ Let  $\pi \in (0, 1]^I$  with  $\sum_{i \in I} \pi_i = 1$  represent the proportion of individuals of each type
- ▶ Given  $\lambda \in \mathbb{R}_{\geq 0}$  and  $J \subseteq I$ , let  $\mathcal{J}^\lambda$  be the set of pairs  $(j, \alpha)$  with  $\sum_{j \in J} \alpha < \lambda \pi_j$ .

## Definition

A  $(\pi, J)$ -sequence of solutions to the optimal investment problem (at time  $s$ , given the individual states and consumption history) consists of a sequence  $(\lambda_n)$  of positive reals with  $\lambda_n \rightarrow \infty$  and a sequence of acceptable solutions  $(\gamma^n)$  for the collective investment problem for  $J$ .

## Definition

The *n-fund approximation* holds if for all  $i \in I$ , all  $\gamma$ , all times  $s$ , all states and the consumption history up to time  $s$  given by  $\gamma$ , we can find a  $(\pi, \{i\})$ -sequence of solutions  $(\gamma_n)$  with

$$\lim_{n \rightarrow \infty} \mathcal{J}_s^{i, \alpha}(\Delta^i(\gamma_n)) \geq \hat{\mathcal{J}}_s^i(\gamma)$$

Subject to minor assumptions, this property can be proved by a tontine to obtain  $\gamma_n$  and then applying the law of large numbers.

# No mutually beneficial contracts with idiosyncratic risk

## Theorem

Suppose

- ▶ The  $\mathcal{J}^i$  are invariant, concave, monotone and satisfy the dominated convergence property
- ▶ The argmax exist so  $\hat{\mathcal{J}}$  is well-defined
- ▶ The  $\hat{\mathcal{J}}$  satisfies the conditions of our Theorem without idiosyncratic risk
- ▶ The  $n$ -fund approximation holds
- ▶  $\gamma^n$  is a  $(\pi, l)$  sequence of solutions to the problem at time 0
- ▶  $\gamma_n \rightarrow \gamma$  almost surely

Then

$$\lim_{n \rightarrow \infty} \mathcal{J}_0^{i, \alpha}(\Delta^i(\gamma_n)) = \sup_{\gamma \in \mathcal{A}^{\{i\}}} \hat{\mathcal{J}}_0^i(\gamma)$$

## Why the premise is reasonable

- ▶ Suppose that  $u$  is a utility function which is increasing, strictly concave, continuously differentiable, satisfies  $u'(0) = \infty$ ,  $u'(\infty) = 0$  and has *asymptotic elasticity* less than 1:

$$AE(u) = \frac{xu'(x)}{u(x)} < 1$$

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The techniques of Kramkov and Schachermeyer (1999) we can find a topology on the space of cashflows such that the set of admissible strategies with utility above a given level is

- ▶ sequentially compact
- ▶ pointwise convergence is equivalent to convergence on this set
- ▶ This implies by the direct approach to calculus of variations that all the required optimization problems have a solution
- ▶ It implies that a convergent subsequence of  $\gamma^{\lambda_n}$  always exists

# Systematic risk

- ▶ Suppose we have a set  $I$  of funds investing in a complete  $(\Omega^M, \mathcal{F}^M, (\mathcal{F}_t^M)_{t \in [0, T]}, \mathbb{P}^M)$  with pricing measure  $\mathbb{Q}^M$
- ▶ Suppose we have an additional space of risk factors  $(\Omega^R, \mathcal{F}^R, (\mathcal{F}_t^R)_{t \in [0, T]}, \mathbb{P}^R)$
- ▶ Suppose we have a preference function  $\mathcal{J}^i$  for and an income stream  $\eta_t^i$  for each  $i \in I$

## Definition

A solution to the collective investment problem consists of

- ▶ A pricing measure  $\mathbb{Q}^R \sim \mathbb{P}^R$
- ▶ A set of processes  $(\gamma_{s,t}^i)_{t \in [0, T]}$  indexed by  $i \in I, s \in [0, T]$  satisfying the causality condition

$$\gamma_{s,t} = \gamma_{t,t} \quad t \leq s$$

and the budget equation

$$\int_0^s \eta_u^i du - \int_s^T \mathbb{E}_{\mathbb{Q}^R \times \mathbb{Q}^M}(\gamma_{s,u}^i \mid \mathcal{F}_s) du \geq 0$$

for all  $i \in I, s \in [0, T]$

such that

- ▶ Each  $\gamma^i$  maximizes  $\mathcal{J}^i((\gamma_{t,t})_{t \in T})$  subject to the budget equation
- ▶ The market clears:  $\sum_{i \in I} \gamma_{s,t}^i$  is  $\mathcal{F}_s^R \times \mathcal{F}_t^M$  measurable

## Numerical Example

- ▶ Suppose  $\Omega^M$  is a Black-Scholes market
- ▶ The  $\Omega^R$  is a 1-dimensional diffusion model for the force of mortality  $\lambda$  for a cohort of investors (similar to a Cairns-Blake-Dowd model).
- ▶ Each fund represents an infinite pool of investors in this cohort, but different Epstein Zin preferences with parameters  $(\alpha, \rho)$
- ▶ Suppose one fund is much larger than the others so their indifference price determines  $\mathbb{Q}$  rather than the clearing condition.

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In this example we can rewrite everything using HJB equations. Because it is a Black-Scholes market which is invariant under scalings of the stock price and because the preferences are positively homogeneous, we can find an ansatz where this has 1 spatial dimension.

## Numerical results (change in certainty equivalent with mutual insurance)

First column is preferences for dominant fund. Row is preferences for small fund.

	$\alpha_1 = -10,$ $\rho_1 = -1$	$\alpha_1 = -5,$ $\rho_1 = -1$	$\alpha_1 = -3,$ $\rho_1 = -1$	$\alpha_1 = -2,$ $\rho_1 = -1$	$\alpha_1 = 3/20,$ $\rho_1 = 1/3$	$\alpha_1 = 1/4,$ $\rho_1 = 1/3$
$\alpha_2 = -10,$ $\rho_2 = -1$	0%	7.76%	22.2%	37.6%	623%	6196%
$\alpha_2 = -5,$ $\rho_2 = -1$	4.96%	0%	1.93%	5.48%	47.4%	92%
$\alpha_2 = -3,$ $\rho_2 = -1$	10.2%	1.41%	0%	0.62%	14%	22.7%
$\alpha_2 = -2,$ $\rho_2 = -1$	13.6%	3.22%	0.5%	0%	5.72%	8.42%
$\alpha_2 = 3/20$ $\rho_2 = 1/3$	21.8%	8.89%	4.32%	2.29%	0%	0.065%
$\alpha_2 = 1/4$ $\rho_2 = 1/3$	21.1%	8.32%	3.87%	1.93%	0.041%	0%

## **Short-term policy:**

- ▶ Collective pension investment can bring real benefits
- ▶ Idiosyncratic longevity risk and heterogeneous investors are easily handled by a tontine
- ▶ In complete markets, the tontine approach is optimal, other claims should be treated sceptically.
- ▶ Pension choice is feasible in collective pensions

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## Long-term outlook:

- ▶ Mutual insurance in incomplete markets is meaningfully possible even if investors only differ by preferences
- ▶ Different ages of investor and wage risk worth considering
- ▶ Potential benefits most likely to be felt by atypical individuals
- ▶ Practical computation of prices is currently an open problem