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Hessian Manifolds

Hessian Manifolds

- Definition of a (locally) Hessian Manifold
- Motivation in information Geometry
- Basic Hessian geometry
- Determining if a manifold is Hessian
- Cartan-Kahler Theory
  - 2-dimensions
  - 4-dimensions
  - Pontrjagin forms in higher dimensions

Definition of a Hessian Manifold

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A *Hessian Manifold* (M, g) is a Riemannian-manifold where we can find a coordinate chart $x : M \to \mathbb{R}^n$ such that, in these coordinates, the Riemannian metric g satisfies

$$g_{ij} = \partial_i \partial_j \phi$$

for some potential function ϕ.

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for some potential function ϕ.

ϕ will automatically be a convex function.

Probability theory recap

Definition

Given a topological space X we generate the *Borel* σ*-algebra*, $\sigma(X)$, of measurable sets using

- Open sets
- Complements
- Countable unions
- Countable intersections

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Definition

A probability measure is a map $\mathbb{P} : \sigma(X) \to [0, 1]$ which is additive on countable disjoint unions and satisfies $\mathbb{P}(X) = 1$

Geometry Example

Example: If M is a smooth manifold and ρ is a density on M we can define:

$$\mathbb{P}_\rho(A) =: \int \mathbf{1}_A \, d\mathbb{P} = \int_A \rho$$

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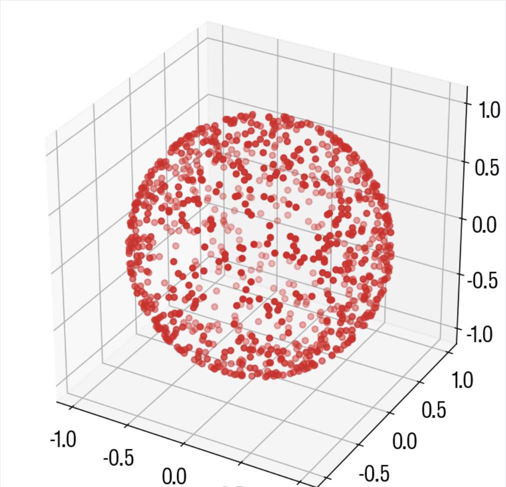

The Radon-Nikodym derivative

Definition

The Diract mass at $x \in X$

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

This represents the deterministic random variable which always takes the value x.

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If $\mathbb{P}_i$ is a countable set of measures and $\lambda_i \in [0, 1]$ are constants with $\sum_i \lambda_i = 1$ then

$$\left(\sum_i \lambda_i \mathbb{P}_i \right)(A) := \sum_i \lambda_i \mathbb{P}_i(A) :$$

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Definition

Let $\rho : X \to [0, \infty)$ satisfy

$$\int_X \rho \, d\mathbb{P} = 1$$

The Hellinger Distance

Let us define a metric on the space of densities. Suppose $\mathbb{P}$ and $\mathbb{Q}$ are densities.

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Choose λ such that

$$\frac{d\mathbb{P}}{d\lambda}, \quad \frac{d\mathbb{Q}}{d\lambda}$$

both exist (e.g. $\frac{1}{2}(\mathbb{P} + \mathbb{Q})$),

Definition

The Hellinger distance is defined by

$$d_H^2(\mathbb{P}, \mathbb{Q}) := \frac{1}{2} \int_X \left(\sqrt{\frac{d\mathbb{P}}{d\lambda}} - \sqrt{\frac{d\mathbb{Q}}{d\lambda}} \right)^2 d\lambda$$

Motivation, the square ensures positivity. The square root ensures independence on choice of λ

Statistical families

Definition

Assume our original topological space, X is a manifold. A *smooth statistical model*, M, is a smooth manifold in the space of densities on X.

Example: The family of normal-distributions on $\mathbb{R}$ is a two-parameter family indexed by $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$ with density given by

$$\frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) d\mathbb{R}.$$

Definition

The Fisher–Rao metric is (up to scale) the Riemannian metric induced by the Hellinger metric.

Example: The family of normal-distributions equipped with the Fisher-Rao metric is (up to a scale factor) isometric to 2-dimensional hyperbolic space.

Exponential families

Definition

Let $F : X \to \mathbb{R}^n$ be a smooth function, λ a reference measure. We say a smooth model M is an *exponential family* w.r.t. λ if it has a smooth chart $\theta : M \to \mathbb{R}^n$ with

$$m = \exp(\theta(m).F(x) - \psi(m))\, \lambda$$

Note that $\psi(m)$ is just a multiplier that ensures the density integrates to 1.

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Example: The family of normal-distributions forms an exponential family w.r.t. the Lebesgue measure

$$\theta = (-\frac{\mu}{\sigma^2}, \frac{\mu}{2\sigma})$$

$$T(x) = (x, x^2)$$

ensures that, for appropriately chosen $\psi : M \to \mathbb{R}$.

$$\frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) = \exp(\eta.T(x) - \psi)$$

Other examples

- Bernoulli
- Binomial with fixed number of trials n
- Poisson
- Negative binomial with known number of failures r
- Pareto with known minimum value x_m
- Weibull with known shape k
- Laplace distribution with known mean μ
- Chi-squared distribution
- Normal distribution with known variance
- Continuous Bernoulli distribution
- Normal distribution
- Log-normal distribution
- inverse Gaussian
- Gamma distrbution
- Inverse gamma distribution
- Generalized inverse gamma distribution
- Scaled inverse chi-squared distribution
- Beta distribution variant 1
- Beta distribution variant 2
- Multivariate normal distribution
- ...

Dually flat manifolds

Theorem

If (M, g) is an exponential family and g is the Fisher-Rao metric then g is Hessian with potential function ψ with respect to the coordinate system θ, i.e.

$$g_{ij} = \partial_i \partial_j \psi$$

- Set $\eta(m) = E_m(F)$. Then η is a *sufficient statistic* for m.
- Define a function $p(m) : X \to \mathbb{R}$ by

$$m = p(m)\lambda$$

- Define

$$\phi(m) = E_m(\log p)$$

Then g is also Hessian with potential function ϕ with resepect to the coordinate system η.

Definition

A Riemannian manifold (M, g) is dually flat if admits a flat, torsion free affine

Questions (Amari)

- Are all metrics g locally Hessian?
- If not, find a tensor which determines whether or not g is locally Hessian.

Solving differential equations – Cartan Kahler theory

Example: Given a symmetric g, when can we locally find a function f and coordinates x such that

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Only if g lies in the n dimensional subspace $\mathrm{Im}\phi \subset S^2 T$ where

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Sometimes we can't find a solution even at a point.

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Since $ddf = 0$ we must have $d\eta = 0$ at x.

Sometimes we can find a solution at a point, but can't extend it even to first order around x.

The Symbol of an operator

- Let E and F be vector bundles and let $D : \Gamma(E) \to \Gamma(F)$ be a differential operator.
- $D : J_k(E) \to F$ where J_k is the bundle of k jets.

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- Define $D_1 : J_{k+1}(E) \to J_1(F)$ to be the first *prolongation*. This is the operator which maps a section e to the one jet of $j_1(De)$.
- Define $D_i : J_{k+i}(E) \to J_i(F)$ to be the i-th prolongation $e \to j_i(e)$

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We can only hope to solve the differential equation $De = f$ if we can find an algebraic solution to every equation

$$D_i e = j_i(f)$$

at the point x.

Applying the fact that derivatives commute may yield obstructions to the existence of solutions to a differential equation even locally.

We have the exact sequence

$$0 \longrightarrow S^k T^* \otimes V \longrightarrow J_k(V) \longrightarrow J_{k-1}(V) \longrightarrow 0$$

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Cartan's Test

Given a basis $\{v_1, v_2, \ldots v_n\}$ for T^*M, define the map:

$$\sigma_{i,m} : S^{k+i}\langle v_1, v_2, \ldots v_m\rangle \otimes V_p \longrightarrow S^i T_p^* \otimes W_p$$

to be the restriction of σ_i.

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Definition

If one can find a basis $\{v_1, v_2, \dots v_n\}$ and a number α such that σ_i is onto for all $i \leq \alpha$ and such that

$$g_{\alpha,n} = \sum_{\beta=0}^{k} g_{\alpha-1\beta}$$

then the differential equation is said to be *involutive*.

Dimension Counting

The dimension of the space of k-jets of 1 functions of n real variables is:

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which is a second order equation for f and coords x. So input is $n+1$ functions of n variables.

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Dimension of space of k jets of g:

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If $n > 2$, d_k^1 grows more slowly than d_k^2. So most metrics are not Hessian metrics.

Curvature of Hessian Manfiolds

Lemma

*Let (M, g) be a Riemannian manifold. Let ∇ denote the Levi–Civita connection and let $\overline{\nabla} = \nabla + A$ be a g-dually flat connection. Then * The tensor A_{ijk} lies in $S^3 T^*$. We shall call it the {S^3*-tensor} of $\overline{\nabla}$. * The S^3-tensor determines the Riemann curvature tensor as follows:

$$R_{ijkl} = -g^{ab} A_{ika} A_{jlb} + g^{ab} A_{ila} A_{jkb}.$$

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- Using metric to identify T^* and T, both $\overline{\nabla}$ and $\overline{\nabla}^*$ are torsion free implies $A \in S^3T^*$

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- Using metric to identify T^* and T, both $\overline{\nabla}$ and $\overline{\nabla}^*$ are torsion free implies $A \in S^3T^*$
- $\overline{R} = 0$. But

$$\begin{aligned}\overline{R}_{XY}Z &= \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X - \overline{\nabla}_{[X,Y]}Z \\ &= R_{XY}Z + 2(\nabla_{[X}A)_{Y]}Z + 2A_{[X}A_{Y]}Z\end{aligned}$$

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- Projecting onto $\Lambda^2 \otimes \Lambda^2$ gives the result.

Curvature obstruction

Define a quadratic equivariant map ρ from $S^3T^* \to \Lambda^2 T^* \otimes \Lambda^2 T^*$ by:

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If g is a Hessian metric R lies in image of ρ.

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In dimension ≥ 5, ρ is not onto. Therefore the condition $R \in Im\,\rho$ is an obstruction to a metric being a Hessian metric.

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$$\dim \mathcal{R} = \dim(\text{Space of algebraic curvature tensors}) = \frac{1}{12} n^2 (n^2 - 1)$$

$$\dim(S^3 T) = \frac{1}{6} n (1 + n)(2 + n)$$

The former is strictly greater than the latter if $n \geq 5$

Lower dimensions

Theorem

In Dimension 3 ρ is onto, so there is no curvature obstruction.

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Pick a random $A \in S^3T^*$ and compute rank of $(\rho*)_A$, the differential of ρ at A. It is 18 whereas the space of algebraic curvature tensors is 20 dimensional. (Proof with probability 1)

i## Finding conditions on the curvature

(Amari) What are the conditions on the curvature tensor for it to lie in the image of ρ?

- This is an *implicitization* question. Im ρ is given parametrically by the map ρ. We want implicit equations on the curvature tensor that define Im ρ.

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- Grobner basis algorithms allow us to solve the latter problem in principle (for fixed n) but not in practice (doubly exponential time is common).
- Algorithms do exist for the real algebraic geometry problem too, but they're even less practical.

A more practical strategy

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- Any invariant linear condition on $\mathcal{R}$ can be expressed as a linear combination of these irreducibles.
- Decompose $S^2\mathcal{R} \oplus \mathcal{R}$ into irreducibles. Any invariant quadratic condition on $\mathcal{R}$ can be expressed as a linear combination of these irreducibles. etc.
- If we have m irreducible components $\rho_1(R)$, $\rho_2(R)$, ..., $\rho_m(R)$. Choose $m+1$ random tensors A and solve the equation

$$\sum_i \alpha_i \rho_i(R) = 0$$

  for α_i to find all linear relations between the components.
- Once the linear relations are known, they can be verified by brute force computer algebra.

A more practical strategy

- The space of algebraic curvature tensors $\mathcal{R}$ is associated to a representation of $SO(n)$.
- Decompose $\mathcal{R}$ into irreducible components under $SO(n)$
- Any invariant linear condition on $\mathcal{R}$ can be expressed as a linear combination of these irreducibles.
- Decompose $S^2\mathcal{R} \oplus \mathcal{R}$ into irreducibles. Any invariant quadratic condition on $\mathcal{R}$ can be expressed as a linear combination of these irreducibles. etc.
- If we have m irreducible components $\rho_1(R)$, $\rho_2(R)$, ..., $\rho_m(R)$. Choose $m+1$ random tensors A and solve the equation

$$\sum_i \alpha_i \rho_i(R) = 0$$

  for α_i to find all linear relations between the components.
- Once the linear relations are known, they can be verified by brute force computer algebra.
- Representation theory of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is simple.