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Amamef25

Robustness of Gamma Hedging

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Joint work with Purba Das (King's College London)

Building on work with Andrei Ionescu (King's College London)

Building on work with Claudio Bellani (Imperial); Damiano Brigo (Imperial); Tom Cass (Imperial)

- Gamma hedging European options - elementary proofs
- Gamma hedging exotics

Mathematical finance without probability

- Föllmer *Calcul d'Itô sans probabilites* 1981
- Bick and Willinger *Dynamic spanning without probabilities* 1994
- Dupire. *Functional Itô Calculus* 2009
- Cont and co-authors e.g.
  - Riga 2015 — analyses continuous time trading strategies
  - Ananova 2020 — draws connections with rough path theory
- Perkowski and Prömel — Vovhk measure and rough path theory 2016
- Allan, Liu, Prömel — includes jumps 2021
- ...

Aim is to give a very simple proof of the effectiveness of the delta- and gamma-hedging strategies.

Remarks

Theorem

Let $\hat{S}_t$ be a path of finite p-variation representing a stock price, for $t \in [0, T]$. Let σ_t be a path of finite q-variation with $p < 3$, $q < 2$ and $\frac{1}{p} + \frac{1}{q} > 1$.
Suppose at each time in $[0, T]$ one can purchase a European option with maturity T and convex, smooth, non-linear payoff function f^1 whose implied volatility is σ_t.
Using the delta-gamma-hedging strategy in the stock, a risk-free asset with return r and this option, one one can replicate any other European option with smooth payoff f^0 and maturity T for the Black-Scholes price: in the sense that the error in the discrete-time hedging strategy tends to 0 as the re-hedging interval tends to zero.

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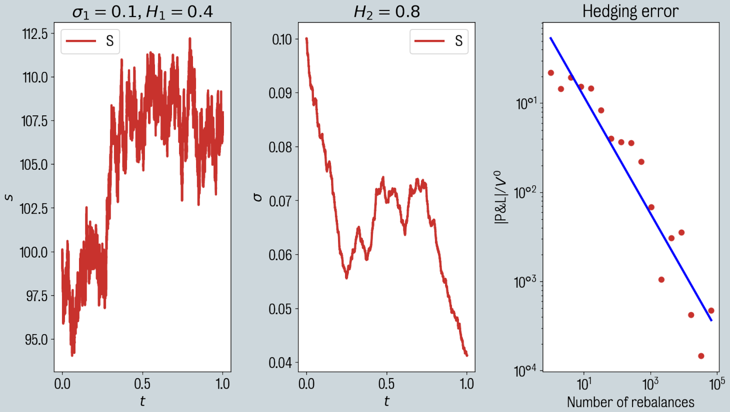
- This is a sure convergence result.
- There are no conditions other than regularity conditions on the trajectory $\hat{S}_t$ or the path $\hat{\sigma}_t$
- Using a change of numeraire we can assume WLOG that $r = 0$ for convenience.

Numerical Example

$$S = S_0 \exp(\sigma_1 W_t^{H_1}), \quad \sigma = \sigma_1 \exp(W_t^{H_2})$$

Hedging a call with strike $K = 100$, maturity $T = 1$ using S and a power option with payoff S_T^2



Prehistoric Smiles

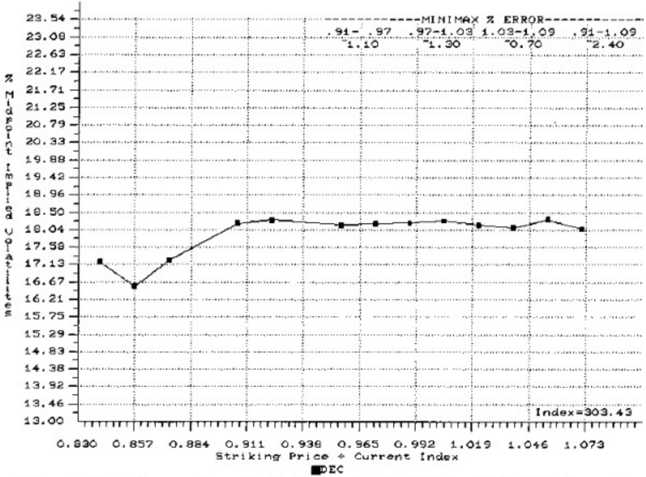


Figure 1. Typical precrash smile. Implied combined volatilities of S&P 500 index options (July 1, 1987; 9:00 A.M.).

Taken from Rubinstein (1994)

Comparison with classical approach

- Traders calibrate a pricing model to market prices, they do not fit a statistical model
- They hedge many Greeks using multiple instruments

- Difference 1: We do not make any assumptions on the underlying dynamics (beyond regularity)
- Difference 2: We will make assumptions on how the market reacts to changes in the underlying
- Equivalently: We do not use a probability model, we use a pricing model.

Our *signal X* consists of both the underlying and information about exchange traded option prices. It represents the data coming from calibrating a model to option prices.

p-th variation

Let $p \in \mathbb{R}_{\geq 1}$ and $\mathcal{V}$ be a real normed vector space. A continuous path $X \in C_0([0,T],\mathcal{V})$ is said to have *vanishing p-th variation* along a sequence of partitions $\pi = (\pi_N)_{N\geq 1}$ if

$$\lim_{N\to\infty} \sum_{[u,v]\in\pi_N} \|X_v - X_u\|^p = 0.$$

Not to be confused with p-variation!!!

Given a path $X \in C([0,T],\mathcal{V})$ we define the p-variation of X by

$$\|X\|_{p-var} = \left(\sup_{\pi} \sum_{[u,v]\in\pi} \|X_v - X_u\|^p \right)^{\frac{1}{p}}.$$

Set up for European Gamma-Hedging

Let π be a sequence of partitions of $[0, T]$ with mesh tending to zero.

- Two vector spaces $\mathcal{V}^1$, $\mathcal{V}^2$
- Two paths $X^1 \in C([0, T], \mathcal{V}^1)$ and $X^2 \in C([0, T], \mathcal{V}^2)$
- X^1 has vanishing p_1-th variation with $1 \leq p_1 \leq 2$.
- X^2 has vanishing p_2-th variation with $1 \leq p_2 \leq 3$
- $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$
- X^1 will represent the slowly varying components of our trading signal, e.g. (t, σ_t)
- X^2 will represent the rough components of our trading signal, e.g. (S_t)

Main gamma-hedging theorem

- $(n+1)$ functions $F^i : \mathcal{V}^1 \times \mathcal{V}^2 \to \mathbb{R}$ which are three-times differentiable
- $(n+1)$ bounded functions $q^i : [0,T] \to \mathbb{R}$

The F^i represent the prices of European options, with index 0 being the option we wish to replicate The q^i will represent the quantities of each option that we hold.

We assume the gamma-hedging conditions

$$\sum_{i=0}^{n} q^i_t \nabla^\alpha F^i = 0, \quad \forall\, \alpha \in \{1,2\}$$

$$\sum_{i=0}^{n} q^i_t \nabla^2 \nabla^2 F^i = 0$$

Then

$$\lim_{N\to\infty} \sum_{[u,v]\in\pi_N} \sum_{i=0}^{n} q^i_u (F^i(X^1(v), X^2(v)) - F^i(X^1(u), X^2(u))) = 0.$$

Proof - apply Taylor's Theorem

$$\sum_{[u,v]\in\pi_N}\sum_{i=0}^{n} q_u^i(F^i(X^1(v),X^2(v)) - F^i(X^1(u),X^2(u))) =$$

$$\sum_{(a_1,a_2)\in\mathcal{I}}\sum_{[u,v]\in\pi_N}\sum_{i=0}^{n} c_{a_1,a_2} q_u^i (\nabla^1_{X_v^1 - X_u^1})^{a_1}(\nabla^2_{X_v^2 - X_u^2})^{a_2} F^i(\xi^{u,N}_{a_1+a_2})$$

Where

- $\mathcal{I} = \{(a_1,a_2) : a_1, a_2 \in \mathbb{Z}_{\geq 0}, 0 < a_1 + a_2 \leq 3\}$
- c_{a_1,a_2} is an appropriate constant
- $\xi_d^{u,N} = X_u$ for $d < 3$
- $\xi_3^{u,N} = X_u + \lambda^{u,N}(X_v - X_u)$ for some $\lambda^{u,N} \in [0,1]$

Observe that all terms vanish in the limit

- For the term $a_1 = 0, a_2 = 3$ use the vanishing third variation of X^2

$$|\sum_{[u,v]\in\pi_N} (\nabla^3_{X^2_v - X^2_u}) F^i)| \leq \sum_{[u,v]\in\pi_N} C \|X^2_v - X^2_u\|^3 \to 0$$

- For the term $a_1 \geq 2$ terms use the vanishing quadratic variation of X^1
- For the term $a_1 = 1, a_2 = 2$ use both vanishing variations and Holder's inequality
- For the term $a_1 = 0, a_2 = 2$ use the gamma-hedging condition

$$\sum_{i=0}^{n} q^i_t \nabla^2 \nabla^2 F^i = 0$$

- For the term $a_1 = 0, a_2 = 1$ use the delta-hedging condition

Corollary: Gamma-hedging with a common PDE

Take $\mathcal{V}^1 = \mathbb{R}$ and $\mathcal{V}^2 = \mathbb{R}^d$ and $X^1(t) = t$

- Let f^i be smooth payoff functions $i = 0, \ldots n$.
- Let $F^i : \mathbb{R}^2 \to \mathbb{R}$ be C^3 functions satisfying a common PDE of the form

$$\nabla^1 F^i = A(t,x)\nabla^2\nabla^2 F^i + B(t,x)\nabla^2 F^i$$

  for operator valued functions A and B
- If X has vanishing p-variation for $p \leq 3$ and

$$\sum_{i=0}^{n} q_t^i \nabla^2 F^i = 0$$

$$\sum_{i=0}^{n} q_t^i \nabla^2\nabla^2 F^i = 0$$

Then

$$\lim_{N\to\infty} \sum_{[u,v]\in\pi_N} \sum_{i=0}^{n} q_u^i (F^i(t, X^2(v)) - F^i(t, X^2(u))) = 0.$$

Example: Gamma-hedging in diffusion models

Let $F^i : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ $(i = 0, \dots n)$ by given by

$$F^i(t_0, x) = E(f^i(S_T))$$

where f^i is a smooth function and S_T is a diffusion process

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t, \quad S_{t_0} = x.$$

Let X^2 be a trading signal. If options are traded at the prices $F^i(t, X^2)$ for $i = 1 \dots d$ then the *gamma-hedging strategy* (where $q_t^0 = -1$), allows us to replicate F^0 so long as X^2 has vanishing p-th variation for $p \leq 3$.

Proof:

By Feynman–Kac the F^i obey a common equation.

Since $q_t^0 = -1$ we rearrange to find

$$F^0(T, X^2) - F^0(0, X^2) = \lim_{N \to \infty} \sum_{[u,v] \in \pi_N} \sum_{i=1}^{n} q_u^i (F^i(t, X^2(v)) - F^i(t, X^2(u))).$$

Example: Gamma-hedging with common maturity

Let $F^i(t, \sigma, S) = BS(f^i, t, T, \sigma, S,)$ be given by Black-Scholes prices with $r = 0$.

Equivalently stated, let σ be the implied volatility process for options with common maturity T.

The delta-gamma hedging strategy allows us to replicate F^0 so long as σ has vanishing $p_1 \leq 2$-th variation, S has vanishing $p_2 \leq 3$-th variation and $\frac{1}{p} + \frac{1}{2} > 1$.

Proof:

Recall that vega is

$$\frac{\partial F}{\partial \sigma}$$

Our main result shows that if we were to delta-vega-gamma hedge then we could replicate the option. The Greeks in the Black-Scholes model satisfy

$$\sigma \tau S^2 \frac{\partial^2 F^i}{\partial S^2} = \frac{\partial F^i}{\partial \sigma}.$$

To prove this note that the pricing kernel satisfies this PDE.

Hence gamma-neutral implies vega-neutral and so it suffices to delta-gamma hedge.

Linear algebra

At each time u write $(dX_u)^a$ for the tensor $(X_v - X_u)^a$ that appear in our sums. View these symbols as basis vectors of a vector space.

- In our main result, we need the coefficients of dX^1_u, dX^2_u and $(dX^2)^2_u$ to vanish.
- When gamma-hedging in a 1-d diffusion model with $X^1 = t$ and $X^2 = S$ we assume the coefficients of dS_u and dS^2_u vanish. The coefficient of dt_u automatically vanishes by the Feynman-Kac equation, so we do not need to "theta hedge".
- When gamma-hedging in model where the options have common maturity, we assume the coefficients of dS_u and dS^2_u vanish. The coefficients of dt_u and $d\sigma_u$ vanish by the Feynman-Kac equation and the relation between gamma and vega.

Example: Delta-hedging in diffusion models

Let $F^i : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ $(i = 0, \dots n)$ by given by

$$F^i(t_0, x) = E(f^i(S_T))$$

where f^i is a smooth function and S_T is a diffusion process

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t, \quad S_{t_0} = x.$$

Let X^2 be a trading signal. If options are traded at the prices $F^i(t, X^2)$ for $i = 1 \dots d$ then the *delta-hedging strategy* allows us to replicate F^0 so long as the q^i are piecewise continuous and

$$\sum_{[u,v] \in \pi_n} \eta_u (X^2_v - X^2_u)^2 \to \sum_{[u,v] \in \pi_n} \eta_u \sigma \sigma^\intercal (v - u)$$

for piecewise continuous $\eta \in \mathcal{V}^\in \otimes \mathcal{V}^\in$. We then say X^2 has quadratic variation given by the intergral of $\sigma\sigma^\intercal$.

Proof:

- Using the notation of our linear algebra arguments, the quadratic variation condition becomes $(dX^2)^2_u = \sigma\sigma^T dt_u$
- This reduces the dimension of the space of coefficients, eliminating the need for gamma hedging

Arbitrage and the Black–Scholes PDE

- If we hedge using instruments whose greeks all satsify some linear relation, then anything we replicate will satisfy the same relations.
- If the greeks of our instruments have maximal dimension, then we have introduced a form of “arbitrage” as we can replicate anything by theta-delta-gamma hedging.
- If there is no “sure-arbitrage”, all instruments must satisfy a common PDE.

Remarks on Smoothness

- If we are hedging using puts and calls, are argument fails if $S_T = K^i$ where K^i is the strike of instrument i.
- If we have a large number of hedging instruments available, we can choose to use the less volatile instruments for hedging.
- We can smooth the payoff and superhedge
- Probabilistic results tell us nothing about null sets

Exotics

- The proof works for hedging barrier options so long as the partial derivatives remain finite
- To hedge Asian options, augment the signal with the running integral of the stock price
- Smooth the payoff to enable superhedging. In a diffusion model, any continuous derivative can be super-hedged for a price arbitrarily close to the risk-neutral price.

Open issue:

- By the Martingale representation theorem all derivatives can be delta-hedged
- Describe which derivatives can be gamma hedged in an elegant fashion
- One criterion: the price can be written as a rough-path integral of the delta, with the gamma being the Gubinelli derivative.

Pedagogical possibilities?

- We have proved that delta-hedging converges without using the Ito integral
- We do not need to introduce the self-financing condition
- There are fewer interpretability issues as our result is a discrete-time result
- There is no need to define geometric Brownian motion using an Ito SDE
- The same proof shows how replication can work in probabilistic and non-probabilistic models

Conclusions

- By Taylor's theorem we can prove the effectiveness of delta-gamma hedging in diffusion models.
- We can also prove the effectiveness of delta hedging when the quadratic variation is known.
- We can understand replication and arbitrage using Taylor's theorem and linear algebra.

Thank you!